Chapter 3
Two Dimensional Strain Analysis
And Hooke’s Law

Screen Titles

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Chapter three covers the definition and analysis of a two dimensional rectilinear state of strain involving both normal and shear components. The topics discussed include development of the equations defining strain components with respect to a rotated axis system, graphical interpretation of rotated axis system strain equations, principal strain and maximum shear strain components, Mohr’s circle construction for two dimensional strain states, Hooke’s law, Poisson’s ratio, two dimensional stress-strain equations, and the relation between elastic and shear moduli. Several sample problems demonstrating the application of the theory are also included.
3. Rectilinear Strain Components

The definitions introduced for normal and shear strain in Chapter 1 are now used to define a two-dimensional state of strain in a body. Consider the deformation of an incremental rectangular element of length \( l_x \) and height \( l_y \). The extension of the edge \( l_x \) is designated \( \delta x \) while the extension of the edge \( l_y \) is designated \( \delta y \). These extensions divided by the length of the edges are designated \( \varepsilon_x \) and \( \varepsilon_y \), the normal strains in the \( x \) any \( y \) directions. Extensional strains are considered positive. In addition to these incremental extensions the element can also undergo a shear deformation that changes the original right angle at the bottom left corner of the rectangular element. The change in this angle is designated the shear strain, \( \gamma_{xy} \). If the angle decreases the shear strain is taken as positive. The two normal strains together with the shear strain define the two-dimensional strain state in the body.

4. Rotated Strain Component - #1

Given a specific two-dimensional strain state relative to an \( xy \) axis system it is desirable to determine the state of strain with respect to a set of axes rotated through some angle \( \theta \) with respect to the original axis system. Each component of the original strain state will contribute to the strains relative to the rotated axis system. Consider the contribution of \( \varepsilon_x \) to \( \varepsilon_a \) measured in the \( a \) direction in the figure. A rectangular element \( dx \ dy \) is chosen such that its diagonal is in the direction \( a \). The element is then given an incremental positive strain \( \varepsilon_x \) resulting in the elongation of \( dx \) by \( \varepsilon_x \ dx \). This gives rise to an extension of the diagonal \( ds \) in the direction \( a \). The magnitude of this extension of \( ds \) due to the \( \varepsilon_x \) extension is determined using the small right triangle in the upper right corner of the extended element \( dx \ dy \). The hypotenuse of the triangle is \( \varepsilon_x \ dx \) so the extension of the diagonal \( ds \) becomes \( \varepsilon_x \ dx \cos\theta \). The strain component in the \( a \) direction due to \( \varepsilon_x \) then becomes \( \varepsilon_x \ dx \cos\theta \). However, \( dx \) over \( ds \) is just \( \cos\theta \) so that the final expression for \( \varepsilon_a \) due to \( \varepsilon_x \) is \( \varepsilon_x \cos\theta^2 \).
5. Rotated Strain Component - #2

The dx dy element from the previous page is now subjected to an incremental positive strain in the y direction, \( \varepsilon_y \). This extends the dy dimension by \( \varepsilon_y \) dy which again increases the length of the diagonal ds that still lies in the a direction. Again the small right triangle in the upper right hand corner is used to determine the amount by which ds is extended. In this instance the hypotenuse of the triangle is \( \varepsilon_y \) dy so that the extension of a is given by \( \varepsilon_y \) dy sine theta. The strain component in the direction of a due to \( \varepsilon_y \) can then be written as \( \varepsilon_y \) dy sine theta divided by ds. But, dy divided by ds is just equal to sine theta. The final contribution to \( \varepsilon_a \) from \( \varepsilon_y \) then becomes \( \varepsilon_y \) sine squared theta.

Rotated Strain Component - #3

The third contribution to \( \varepsilon_a \) comes from gamma xy, the shear strain relative to the xy axes. The development begins again with the element dx dy with diagonal ds in the a direction. An incremental positive shear strain gamma xy is applied to create a rotation of the dx edge while dy remains as it was originally. For a small angle of rotation the hypotenuse of the right triangle in the upper right can be expressed as gamma x dx. Thus the extension of ds becomes gamma xy dx times sine theta. The strain component in the direction of a due to gamma xy can then be written as gamma xy dx times sine theta divided by ds. Since dx over ds is just cosine theta the final expression for the contribution of gamma xy to \( \varepsilon_a \) becomes gamma xy times sine theta cosine theta.
7. Total Rotated Strain (direction a)

The three xy strain state contributions from the previous pages are now added together to give the normal stain epsilon a in the a direction.

The equation for epsilon a becomes epsilon x cosine squared theta plus epsilon y sine squared theta plus gamma xy sine theta cosine theta. It is convenient to rewrite this equation into double angle form using the trig identities indicated. This gives the final expression epsilon a equal to the quantity epsilon x plus epsilon y divided by 2 plus the quantity epsilon x minus epsilon y divided by 2 times cos two theta plus the quantity gamma xy divided by 2 times sine two theta. Does this look familiar? It should, recall the development of a similar equation for a two-dimensional state of stress.

8. Total Rotated Strain (direction b)

The magnitude of the normal strain in direction b, the second strain component for the rotated rectilinear axis system shown in the figure, is now desired. This can be obtained from the epsilon a equation by substituting theta equal to the quantity theta plus pi over 2 into the equation for epsilon a. Do this as an exercise to show that the equation for epsilon b at the bottom of the page is obtained. Click on the solution button to check the procedure for getting this result or go on to the next page.

(Solution on Page 71)
9. Rotated Shear Strain Component - #1

A similar development will now be undertaken to obtain an expression for the shear strain gamma ab relative to the ab axis system in terms of the strain state specified with respect to the xy axis system. For this purpose an elemental rectangle of dx' dy is added to the element dx dy. The dimension dx' is chosen such that its diagonal ds' is perpendicular to ds the diagonal of the element dx dy. The combined element is subjected to a incremental normal strain epsilon x. The right side of the rectangular element extends epsilon x dx while the left side extends epsilon x dx'. This results in extensions of both ds and ds' giving rise to two parts to the shear strain component gamma ab due to epsilon x. These components are represented by the change in angle of ds and ds'. The change in the angle of ds is the distance A A' divided by ds. This is positive since the rotation tend to make the angle between ds and ds' smaller. The change in angle contribution due to the rotation of ds' is the distance B B' divided by ds'. This contribution is negative as the change tends to increase the angle between ds and ds'. From the triangle in the upper right the distance A A' is just epsilon x dx cosine theta while the distance B B' from the right triangle in the upper left is epsilon x dx' sine theta. Since dx over ds is cosine theta and dx' over ds' is sine theta the final expression for gamma ab due to epsilon x is gamma xy times the quantity cosine squared theta minus sine squared theta.

10. Rotated Shear Strain Component - #2

To develop the contribution to gamma ab from a normal strain in the y direction an incremental strain epsilon y is applied to the elemental rectangle used on the previous page. This extends the upper edge by an amount epsilon y dy which again results in small rotations of the diagonal ds and ds'. The change in the angle of ds is again positive and is given by the distance A A' divided by ds. The change in the angle of ds' is also positive and can be expressed as B B' divided by ds'. The distance A A' from the small upper right triangle is epsilon y dy cosine theta while B B' from the upper left right triangle is epsilon y dy sine theta. With dy over ds equal to sine theta and dy over ds' equal to cosine theta the final expression for gamma ab due to epsilon y is given by 2 epsilon y sine theta cosine theta.
11. Rotated Shear Strain Component - #3

An incremental shear strain component \( \gamma_{xy} \) is now applied to the rectangular element to develop its contribution to \( \gamma_{ab} \). The lower edge of the rectangle is rotated about the origin by the angle \( \gamma_{xy} \) while the vertical dimension remains parallel to the y-axis. This gives rise to a positive change in the angle of \( ds \) and a negative change in the angle of \( ds' \). The change in angle of \( ds \) is given by \( \frac{A A'}{ds} \) while the angle change in \( ds' \) can be written as \( \frac{B B'}{ds'} \). From the upper right triangle \( A A' \) is equal to \( \gamma_{xy} dx \cos \theta \) and \( B B' \) from the upper left triangle is given by \( \gamma_{xy} dx \sin \theta \). Again recognizing that \( dx \) over \( ds \) is \( \cos \theta \) and \( dx \) over \( ds' \) is \( \sin \theta \) the final expression for the \( \gamma_{ab} \) contribution from \( \gamma_{xy} \) is \( \gamma_{xy} \times (\cos^2 \theta - \sin^2 \theta) \).

12. Total Rotated Shear Strain

The three \( \gamma_{xy} \) strain state contributions from the previous pages are now added together to give the total shear strain \( \gamma_{ab} \) relative to the rotated ab axis system. The equation for \( \gamma_{ab} \) becomes:

\[
\gamma_{ab} = 2(\varepsilon_y - \varepsilon_x)\sin \theta \cos \theta + \gamma_{xy} (\cos^2 \theta - \sin^2 \theta)
\]

Substitute:

\[
\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta, \quad \cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta
\]

\[
\gamma_{ab} = (\varepsilon_y - \varepsilon_x)\sin 2\theta + \gamma_{xy} \cos 2\theta
\]

This should again look familiar recalling a similar looking equation for shear stress relative to a rotated axis system in the study of two dimensional stresses.
13. Total Rotated Strain Equations

Listed below are the three equations for determining all components of a two-dimensional state of strain measured relative to an ab rotated axis system in terms of the specified two-dimensional components of strain relative to some xy axis system and the angle of rotation to the new ab directions. These three equations are in effect exactly the same as the equations for a two-dimensional stress state with respect to a rotated axis system except for one small difference. What is that difference? Check on the compare button to see if your answer is correct.

(Comparison on Page 71)

14. Graphic Interpretation

Since the rotated axis strain equations are effectively the same as the rotated axis stress equations they can be given a similar graphical interpretation and representation. By designating epsilon avg as the quantity epsilon x plus epsilon y over 2 and eliminating the angle theta between two equations on the previous page an equation is obtained which describes a circle in the epsilon and gamma over two axis system whose center is at epsilon average on the epsilon axis as shown on the left. As indicated on the right the radius of the circle is given by R which is equal to the quantity epsilon x minus epsilon y over 2 quantity squared plus gamma xy over two squared. From the circle it is seen that at some rotated axis position the normal strains will have maximum and minimum values together with a shear strain of zero. There also exist a maximum value of shear strain relative to a set of axes for which the normal strains are equal.
15. Principal Strain Equations

From the circle representation on the previous page equations can now be written for the maximum and minimum values of normal strain and the maximum shear strain for any given two-dimensional strain state. Note that the maximum and minimum normal strains only differ by the sign of the second term. Also the value of the maximum shear strain is just equal to the second term in the max and min normal strain equations.

\[ \varepsilon_{\text{max}} = \varepsilon_{\text{avg}} + R = \left( \frac{\varepsilon_x + \varepsilon_y}{2} \right) + \sqrt{\left( \frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \left( \frac{\gamma_{xy}}{2} \right)^2} \]

\[ \varepsilon_{\text{min}} = \varepsilon_{\text{avg}} - R = \left( \frac{\varepsilon_x + \varepsilon_y}{2} \right) - \sqrt{\left( \frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \left( \frac{\gamma_{xy}}{2} \right)^2} \]

\[ \left( \frac{\gamma_{xy}}{2} \right)_{\text{max}} = R = \sqrt{\left( \frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \left( \frac{\gamma_{xy}}{2} \right)^2} \]

16. Mohr Circle for Strain

A Mohr circle for a two dimensional strain state can be constructed and used in a similar manner to that presented for a two-dimensional stress state in chapter 2. The construction is begun by establishing an epsilon and gamma over two coordinate system. In this coordinate field two sets of points are plotted. The first is epsilon x and gamma xy over two. The second set is epsilon y and minus gamma xy over two. A line connecting these two sets of coordinates is the diameter of the circle. All states of strain relative to rotated axes now lie as coordinates at the end of some diameter of this circle. To establish this diameter for the rotated axis system x'y' an angle of two theta is measured from the original diameter in the direction opposite to the rotation theta for the real axes. Where this new diameter intersects the circle are the coordinates epsilon x' and gamma x'y' over two at one end and epsilon y', minus gamma x'y' over two at the other end.
Sample Problem - 1

Given that:
\( \varepsilon_x = 500 \times 10^{-6} \text{ in./in.} \)
\( \varepsilon_y = 140 \times 10^{-6} \text{ in./in.} \)
\( \gamma_{xy} = 360 \times 10^{-6} \text{ in./in.} \)

Determine:
1. the values of \( \varepsilon_a, \varepsilon_b, \gamma_{ab} \) for an "a" axis
   60 degrees counterclockwise from the "x" axis.
2. the values of the principal strains and the maximum shear strain.

17. Sample Problem - 1

In this sample problem the values of a two-dimensional strain state are given for an xy axis system. In part a determine the values of epsilon a, epsilon b and gamma ab for an axis system rotated 60 degrees counterclockwise from the x axis. In part b determine the values of the principal strains and maximum shear strain for the given two-dimensional state of strain. After working out results for both parts check your process of solution and results by clicking on the solution button.

(Solution on Pages 72 and 73)

Sample Problem - 2

Given that:
\( \varepsilon_x = 500 \times 10^{-6} \text{ in./in.} \)
\( \varepsilon_y = 140 \times 10^{-6} \text{ in./in.} \)
\( \gamma_{xy} = 360 \times 10^{-6} \text{ in./in.} \)

Determine:
1. the orientation of the axes of principal strain relative to the xy axes.
2. the orientation of the axes of maximum shear strain relative to the xy axes.

18. Sample Problem – 2

For the same given strain state as in sample problem 1 determine for part a the orientation of the axes of principal strain relative to the xy axis system. In part b determine the orientation of the axes of maximum shear strain relative to the xy axis system. Click on the solution button to check your results and solution process.

(Solution on Page 74)
19. 45 Degree Strain Rosette

If the normal strains are known in three directions, which are separated by angles of 45 degrees this, is sufficient information to establish the Mohr strain circle. This is useful when a 45-degree strain gauge is used to measure stain in an actual loaded member. The construction is begun by setting up an epsilon, gamma over two coordinate system. Epsilon 1, epsilon 2 and epsilon 3 are marked off along the epsilon axis. The center of the circle is then located at epsilon 1 plus epsilon 3 over two. The distance between the center and the epsilon two value which is the quantity epsilon 1 plus epsilon 3 minus two epsilon 2 all over 2 is determined and measured vertically up at the location of epsilon 1. This vertical distance corresponds to the value of gamma 13 over 2. Hence this defines one end of the diameter of the circle. The other end is of course epsilon 3, minus gamma 13 over 2. With this diameter the circle can be drawn. A line drawn from where epsilon 2 intersects the circle and it center represents on half of the diameter for the axis system defined by the direction of epsilon 2 and an axis perpendicular to it. Note that the construction results in angles of two theta of 90 degrees between the diameter for axes 13 and the direction of epsilon three, which agrees with the fact that on the rosette axes 1,2 and 3 are separated by 45 degrees.

20. Hooke’s Law

The page on Hooke’s law from Chapter 1 is repeated here to introduce a brief overview of how two-dimensional strain and stress states are related through material properties. This law of elasticity simply states that for most materials stress and strain are linearly related for small values of strain. The constant of proportionality for normal stress and strain is the modulus of elasticity, E, sometimes called Young's modulus. For shear stress and strain the constant of proportionality is designated G and is called the shear modulus. To relate two dimensional stress and strain states a third material property covered on the next page is required.
21. Poisson’s Ratio

Consider the physical deformation behavior of an incremental element \( dx \, dy \) of a real material subjected to an extensional strain \( \epsilon_x \). The \( x \) dimension of the element elongates by an amount \( \epsilon_x \, dx \). At the same time the \( y \) dimension of the element contracts by an amount that is proportional to the \( x \) direction elongation and is expressed mathematically by the expression \( \nu \, \epsilon_x \, dx \). The parameter \( \nu \) is a constant of proportionality called Poisson’s ratio after a French mathematician and is a material property like the two moduli \( E \) and \( G \). It is defined as the ratio of unit lateral contraction to unit axial elongation. Thus \( \epsilon_y \) for this loading and deformation state is just \( \nu \) times \( \epsilon_x \). The value of \( \nu \) for real materials always lies between zero and one half.

22. Strains from 2 D Stress State

Now consider the same incremental element subjected to both \( \sigma_x \) and \( \sigma_y \) stresses at the same time. The \( \epsilon_x \) strain will now consist of two terms. The first is the direct elongation due to the \( \sigma_x \) stress or simply \( \sigma_x / E \). The second term will be a contraction in the \( x \) direction due the Poisson’s ratio effect of the applied \( \sigma_y \) stress. It is expressed as \(-\nu \, \sigma_y / E\). Added together they give the total strain in the \( x \) direction as indicated. In a similar fashion the strain \( \epsilon_y \) in the \( y \) direction is given by the same equation with \( \sigma_x \) and \( \sigma_y \) reversed. Now write out the three equations for \( \epsilon_x, \epsilon_y \) and \( \epsilon_z \) for a cubical element subjected to the three normal stresses \( \sigma_x, \sigma_y \) and \( \sigma_z \). Also for the two dimensional state of strain develop equations for \( \sigma_x \) and \( \sigma_y \) in terms of \( \epsilon_x, \epsilon_y \) and \( \nu \). Check your answers by clicking on the solution button.

(Solution on Page 75)
Sample Problem - 3

A cylindrical rod of length L and radius R is subject to a normal axial stress. Assuming small strains show that for the volume change to remain positive the value of Poisson's ratio can not exceed 1/2. Hint: Write an expression for the change in volume in terms of the deformed volume minus the original volume assuming an axial strain of epsilon. Do not forget that the radius will decrease due to Poisson's ratio and that the axial strain is small.

23. Sample Problem – 3

The purpose of this problem is to demonstrate why the value of nu cannot exceed one half for real materials. The small strain assumption means that squared and cubed strain terms can be neglected relative to first order strain terms. Click on the solution button to check your result for the change in volume.

(Solution on Page 76)

Pure Shear Stress and Strain States

The next two pages will be used to develop a relationship between E, G and nu based on Hook's law of deformation behavior. This development is begun by considering a state of pure shear stress, Tau, in two dimensions. The Mohr circle for this stress state is shown directly below the element on which the shear stresses act. It is observed that the principle stress values, sigma 1 and sigma 2, are simply plus and minus Tau. From the Mohr strain circle below the element on which the principal stresses act it is observed the value of maximum shear strain is just twice epsilon 1 or minus epsilon 2.
25. Relation Between E and G

Epsilon 1 is now written in terms of sigma 1 and sigma 2 both of which can then be replaced by Tau with consideration of sign. A second equation for epsilon 1 is written in terms of gamma max over two, which can be replaced by Tau over 2G. The two equations for epsilon 1 are now set equal to one another and the common Tau term is divided out. This is then rearranged to give the expression G equal to E divided by 2 times the quantity one minus two nu. Hence G can be calculated if E and Nu are known. Also since Nu is limited between zero and one half this in turn means that G must lie between one third and one half E. Note how this works for steel that has an E of 30 million psi and a G of 12 million psi.

26. Review Exercises

In this exercise the items in the list on the left are to be matched with the most appropriate phrase on the right. Place the cursor over an item on the left and hold down the left button. A pencil will appear that can be dragged to one of the green dots on the right. If the right choice is made the arrow will remain. If the selection is incorrect the arrow will disappear. After the exercise is completed proceed to the next page.

(Solution in Appendix)
27. Off Line Exercises

This off line exercise is a practical problem that requires using materials covered both in Chapter 3 and Chapter 2. In addition to determining the specific numerical values requested it is suggested that the appropriate Mohr circles be drawn to check your calculated values. If you can successfully complete this problem you will have a good understanding of two-dimensional stress and strain states and their relation to one another. Good luck.

Off Line Exercise

A 45° rosette strain gauge used to determine the stress in a loaded steel beam gives the following readings:

\[ \varepsilon_\theta = 300 \times 10^{-6} \text{ in./in.} \]
\[ \varepsilon_{45^\circ} = 100 \times 10^{-6} \text{ in./in.} \]
\[ \varepsilon_{90^\circ} = -50 \times 10^{-6} \text{ in./in.} \]

Determine:

a. The magnitude and direction of the principal stresses.

b. The value of the maximum shear stress and orientation of the planes on which it acts.
Chapter 3

Two Dimensional Strain Analysis

Problem Solutions

Screen Titles

- Total Rotated Strain (direction b)
- Rotated Stress Equations
- Rotated Normal Strains
- Rotated Shear Strain
- Principal Strains
- Maximum Shear Strain
- Generic Principal Axis Direction
- Principal Direction Calculation
- 3D Strain Equations
- Stress in terms of strain
- Volume Calculations
- Simplifying Results
1. Total Rotated Strain (direction b)

Begin with the equation for epsilon a with theta plus Pi over 2 substituted for theta. Now cosine of two times the quantity theta plus pi over two is equal to minus cos two theta and sine of two times the quantity theta plus pi over 2 is equal to minus sine two theta. Thus the equation for epsilon b contains the same three terms as the equation for epsilon a with the exception that the second and third terms are now negative. Click on the return button to go back to the next page in chapter 3.

2. Rotated Stress Equations

The three listed rotated axis stress equations will look exactly like the rotated axis strain equations if the normal stresses are replaced by normal strains and the shear stresses are replaced by shear strains divided by two. In other words sigma x', y' become epsilon a', b' and sigma x, y become epsilon x, y together with tau ab' replaced by gamma xy' over two and tau xy replaced by gamma xy over two. Click on the return button to go back to the next page in chapter 3.
### 3. Rotated Normal Strains

Substituting the given values of epsilon \( x \), epsilon \( y \) and gamma \( xy \) into the equation for epsilon \( a \) together with two theta equal to a positive 120 degrees and carrying out the indicated mathematical manipulations gives a final value for epsilon \( a \) of 386 micro inches per inch. Epsilon \( b \) is similarly determined by simply applying the appropriate sign changes for the last two terms in the epsilon \( a \) equation. This gives a final value of 254 micro inches per inch.

### 4. Rotated Shear Strain

Again the given strain state values along with two theta equal to 120 degrees are substituted into the equation for gamma \( ab \) and the required mathematical manipulations are carried out. This gives a final value for gamma \( ab \) of minus 492 micro inches per inch. Shown to the right, approximately to scale, is the Mohr Strain circle with the original and rotated axis strain values. Note that the negative rotation on the circle of 120 degree results in a negative value of gamma \( ab \) associated with epsilon \( a \) as specified by the numerical determination. Click on the return button to go back to the next page in chapter 3.
5. Principal Strains

The given xy strain state from the previous problem are now substituted into the equation for the maximum normal strain. Carrying out the indicated mathematical manipulations results in epsilon max equal to 569 micro inches per inch. Since the minimum principal strain only changes by the sign of the second term in epsilon max the numerical result simply becomes 71 micro inches per inch.

\[
\varepsilon_{\text{max}} = \frac{\varepsilon_x + \varepsilon_y}{2} - \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \frac{\gamma_{xy}}{2}}
\]

\[
= \left(\frac{500 + 140}{2}\right) + \sqrt{\left(\frac{500 - 140}{2}\right)^2 + \left(\frac{300}{2}\right)^2} \times 10^{-6}
\]

\[
= (320 + 249) \times 10^{-6} = 569 \times 10^{-6} \text{ in./in.}
\]

\[
\varepsilon_{\text{min}} = (320 - 249) \times 10^{-6} = 71 \times 10^{-6} \text{ in./in.}
\]

6. Maximum Shear Strain

The maximum shear strain is determined using the equation for gamma max over two. This is just the last term in the principal stain equations, which gives a final value of 498 micro inches per inch. Again the Mohr circle for strain from the previous problem is shown on the right with the principal strains and maximum shear strain indicated. Note that they also seem to be correct values as indicated by the approximate scale of the circle. Now click on the return button to go back to the next page in chapter 3.
7. Generic Principal Axis Direction

To determine the direction of the principal axes attention is directed to the generic Mohr circle for strain shown on the left. The highlighted red triangle can be used to graphically define the tangent of two theta that represents the rotation of the initial strain state circle diameter defined by epsilon x and gamma xy over two to the horizontal epsilon axis that defines the principal strain orientation. From the geometry of the figure it is seen that the height of the triangle is simply gamma xy over two while the base is equal to epsilon x minus epsilon a average, that is epsilon x plus epsilon y over 2, giving the result epsilon x minus epsilon y over 2. Thus the tangent of two theta that defines the orientation of the principal strain axes becomes gamma xy divided by the quantity epsilon x minus epsilon y.

8. Principal Direction Calculation

The appropriate xy axis strain state values are substituted into the equation for tangent two theta. Since tangent of two theta is one then two theta is 45 degrees which makes theta just 22.5 degrees. Since two theta is clockwise on the Mohr circle for strain in the previous problem the rotation of the principal axes will be counterclockwise in the real physical system shown as indicated in the figure on the right. The axis relative to which the maximum shear strain acts is shown in red at 45 degrees counterclockwise from the epsilon max direction. Click the return button to go to the next page in chapter 3.
9. 3D Strain Equations

The Poisson’s ratio effect due to the sigma z stress must be added to the two-dimensional strain equation for epsilon x. The added term is given by minus nu times sigma z divided by E assuming that sigma z is positive. This same term must also be added to the epsilon y strain equation in two dimensions to include the Poisson’s ratio effect of the sigma z stress. Using the form of the two expanded strain equations for epsilon x and epsilon y a similar equation can be written for the strain in the z direction. It too will consist of three terms. The first is the direct effect of the sigma z stress while the second two terms are the Poisson’s ratio effect of the sigma x and sigma y stresses.

10. Stress in terms of strain

An equations for the sigma y stress in terms of normal strains in two dimension is obtained by eliminating the sigma x stress between the two strain equations. This is easily accomplished by multiplying the epsilon x equation by nu and adding it to the equation for epsilon y. The result is an equation that only contains the stress sigma y and the two normal strain equations. Solving for sigma y gives E over the quantity one minus nu squared multiplied by the quantity epsilon y plus nu times epsilon x. A similar equation can be developed for sigma x in terms of epsilon x and epsilon y. Click the return button to go to the next page in chapter 3.
11. Volume Calculations

The volume of the unloaded cylindrical rod is simply \( \pi R^2 \) squared times the length, \( L \). Applying an axial normal stress with a corresponding axial strain of \( \varepsilon \) to the rod results in a new length expressed as \( L(1 + \varepsilon) \) times the quantity one plus \( \varepsilon \). At the same time the radius of the deformed rod becomes \( R(1 - \nu \varepsilon) \) due to Poisson’s ratio contraction from the effect of the axial stress. Thus the deformed volume becomes \( \pi R^2 (1 - \nu \varepsilon)^2 L(1 + \varepsilon) \) times the quantity one minus \( \nu \varepsilon \) times \( \varepsilon \). The change in volume defined by the deformed volume minus the original volume can then be expressed by the equation at the bottom of the page.

12. Simplifying Results

The bracketed terms on the right side of the delta volume divided by \( \pi R^2 L \) equation are now multiplied out. Canceling out the ones and neglecting all higher order \( \varepsilon \) terms due to the small value of the strain gives the final result that the change in volume term is just equal to \( \varepsilon \) times the quantity one minus \( 2\nu \). If \( \nu \) become greater than one half the change in volume of the rod will become negative which is not possible under the action of an axial elongation. Hence the value of Poisson’s ratio is limited physically to one half.