Chapter 10

Limit Loading

Screen Titles

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Geometry of Bending
Strain and Stress Distributions
Equilibrium Conditions
Stress-Strain Relations
Axial Equilibrium
Moment Equilibrium
Plastic Moment
Alternate Solution
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Review Exercise
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Chapter 10 introduces the concept of limit loading and applies the theory to a number of practical problems. The subject of limit loads and their prediction addresses the fact that much of the static load carrying capacity of mechanical elements produced from a reasonably ductile material resides in its properties and behavior beyond the elastic limit. Based on an idealized behavior for a ductile material this chapter develops a model process of solution that permits the determination of limit loads on mechanical beam elements subjected to transverse bending. A number of simple problems are used to demonstrate the application of the solution process. This includes statically indeterminate problems and beams of curved geometry. Several exercises are also included to provide the reader with practice in the application of the solution process.

Listed on this page are all the individual pages in Chapter 10 with the exception of the exercise problems. Each page title is hyperlinked to its specific page and can be accessed by clicking on the title. It is suggested that the reader first proceed through all pages sequentially. Clicking on the text button at the bottom of the page provides a pop up window with the text for that page. The text page is closed by clicking on the x in the top right corner of the frame. Clicking on the index button returns the presentation to the page index of chapter 10.
3. Basis of Limit Load Analysis

Virtually all mechanical element stress analysis models assume a linear stress strain behavior for the material. Once the internal stresses created by the external loads are determined the various theories of static strength and fatigue are applied to determine the acceptability of the design. In static strength determination this approach neglects the great potential for a material to carry an external load that exists beyond the small range of its elastic behavior. For example consider how small the area representing stored energy is under the linear portion of the stress strain diagram as compared to the material’s capacity to absorb work and energy as depicted under the plastic behavior portion of the stress strain diagram. Once the yield strain is exceeded most reasonably ductile materials can continue to deform and experience strains of several hundred times the yield strain before fracture takes place. Estimating the limit loads associated with this extended behavior is the subject of this chapter.

4. Geometry of Bending

The developments to follow will emphasize determining the limit loads for beam type elements subject to bending by transverse loading. From experimentation it is well established that plane cross sections in a beam remain plane after bending. Thus, two parallel cross sections before bending will be inclined with respect to each other when the beam is subjected to curvature. Thus, longitudinal element fibers at the top of the beam will undergo compression while element fibers at the bottom surface are extended. Hence, there exists some location through the thickness of the beam where there is no strain applied to a longitudinal element. This location is referred to as the neutral axis of the beam. The extensional strain of a fiber at some distance y below the neutral axis can be determined using the geometry of the bent beam on this page. The strain epsilon is equal to the distance from the neutral axis to the fiber of interest divided by the radius of curvature of the beam. Note that this results in compressive strains above the neutral axis and positive tensile strains in the plus y direction.
5. Strain and Stress Distributions

The linear strain distribution model from the previous page results in some form of normal stress distribution across the depth of the beam. This distribution may be linear as for an elastic material or in the most general case it may be non-linear as also depicted on this page. Irrespective of the form of the distribution the resultant force and moment of this stress distribution must satisfy the effect of the external loading at that cross section. If the beam is subjected to only transverse loading the axial force must be zero and the moment of the stress distribution about the neutral axis must be equal to the bending moment at that location.

6. Equilibrium Conditions

The resultant axial force equilibrium condition is expressed as the integral of the stress over the area of the cross section. For simplicity the cross section will be taken to be rectangular of width \( b \) and height \( h \). Thus, the integral of \( \sigma \) da over the total area can be written as \( bh \) times the integral of \( \sigma \) dy over the height. Making use of the expression that the strain is equal to \( y \) over \( \rho \) the integral can then be written as \( \sigma \) d\( \epsilon \) from \(-\epsilon_2\) to \(\epsilon_1\). The moment equilibrium condition is initially expressed as the integral of the product of \( \sigma \) and \( y \) over the cross section. This can then be rewritten for the rectangular section as \( bh \) times the integral of \( \sigma \) dy from \(-h_2\) to \(h_1\). Again substituting the relationship between \( \epsilon \) and \( y \) gives finally \( \rho^2bh \) times the integral of \( \sigma \) d\( \epsilon \) from \(-\epsilon_2\) to \(\epsilon_1\) all equal to the bending moment \( M \).
7. Stress Strain Relation

To evaluate the equilibrium integrals on the previous page it is now necessary to assume some representation for the stress strain relationship for the material. A simple, yet reasonable model, depicting the behavior of an idealized ductile material, is now assumed. As shown in the figure this model assumes that the yield stress in tension and compression is equal in magnitude and that the tensile modulus and compressive modulus in the elastic region are equal. Once the yield stress is reached it is further assumed that the stress remains constant independent of strain and that the strain is unlimited. Although idealized this model has proven to be quite useful in limit load analysis.

8. Axial Equilibrium

The model from the previous page is substituted into the axial force equilibrium equation. This gives rise to three separate integrals that must be evaluated. Note that the stress in the second integral is expressed in terms of the linear relationship that it has with the strain in terms of the modulus in the elastic region. Evaluating the integrals and combining the results gives a final equation that indicates epsilon one must be equal to epsilon two for the rectangular cross section. This in turn implies that h1 must be equal to h2, which further implies that the neutral axis must pass through the centroid of the cross section just as in simple elastic beam theory.
9. Moment Equation

Substituting the idealized ductile stress strain relationship into the moment equilibrium equation again results in three distinct integrals. This set of integrals lead to quadratic and cubic terms in strain. Recognizing that epsilon one is equal to epsilon two from the previous page and expressing the modulus as the yield stress, sigma y, divided by the yield strain, epsilon y, a final relationship for the moment divided by b rho squared is obtained as sigma y times the quantity epsilon one squared minus one third epsilon y squared. As indicated earlier and expressed in terms of the idealized stress strain relationship epsilon y is significantly smaller than epsilon one so that its contribution can be neglected. Recall that it was earlier pointed out that the total strain that can be sustained by a ductile material can be several hundred times the yield strain. Thus the final equation for the bending moment is taken to be sigma y times b rho squared. The task that now remains is to relate this result to linear beam theory.

10. Plastic moment

The moment expression from the previous page is designated the plastic moment indicated by the sub p. The epsilon rho term is eliminated by substituting h over two rho for epsilon one. The plastic moment then becomes one-fourth sigma y times b h squared. This will now be compared to the moment in an elastic beam when the maximum stress is equal to sigma y. From the familiar formula that sigma is equal to M c over I the moment at the inception of yielding can be written as sigma y times I over c. For a rectangular cross section I is simply b h cubed over twelve and c is h over two. Substituting I and c into the Me expression gives one-sixth sigma y times b h squared. It is seen finally that the ratio of the plastic moment to the elastic moment is three halves or 1.5. This indicates that that moment that initiates yielding can be increased by 50 % before the cross section has become completely plastic as defined by the assumed material model.
Alternate Solution

The final analytic result for the magnitude of the plastic moment could have been obtained more easily with the model shown on this page. Assuming that the neutral axis does pass through the centroid of the cross section and the stress distribution is constant over the cross section at \( \sigma_y \) then the resultant force above and below the neutral axis is just \( \sigma_y \) times half the area of the section. These two resultant forces are equivalent to a couple of magnitude \( F_1 \) times \( h \) over two, which in fact is the bending moment on the section. Hence \( M_p \) becomes \( \sigma_y \) times \( bh^2 \) squared over four as determined earlier. This method can be used to very quickly determine the magnitude of the plastic moment for any cross sectional shape provided it is assumed that the stress above and below the neutral axis is equal to a constant value, \( \sigma_y \).

Exercise Problem - 1

In this exercise you are asked to determine an expression for the plastic moment in a solid circular cross section using the technique demonstrated on the previous page. Compare this result with the elastic moment for the same cross section. That is, the moment that first initiates yielding in the section. You will find that the ratio of the plastic moment to the elastic moment is greater than for a rectangular cross section. Can you explain why this is so? When you have completed your solution click on the solution button to check your result before proceeding further in the chapter.

(Solution on Page 241)
13. Plastic Hinge Behavior

The top figure on this page shows a beam fixed at one end, simply supported at the other and loaded with a concentrated force along its length. This is a statically indeterminate problem as the reactions $F_1$, $F_2$ and the moment, $M$, cannot be determined from the equations of equilibrium alone. However, it is possible to possible to sketch what the bending moment diagram will look like shown directly below the drawing of the beam. The curvature of the beam is also shown as would be anticipated providing that elastic behavior is taking place. Keep in mind that the scale of the indicated displacements are very exaggerated compared with the length of the beam. Now consider what happens as the load $F$ is increased. The magnitude of the bending moment will increase at both the location of the load and at the fixed end. Assuming the idealized stress strain behavior assumed earlier for a ductile material the maximum bending moment that can be achieved at these two points will be the plastic moment for the cross section of the beam. When this occurs the material in the beam at these locations can undergo extensive yielding so the beam appears to be hinged at these points with the remainder of it appearing straight since the elastic deformations away from these points will be very small. These two locations are referred to as plastic hinges and can be used to determine the magnitude of the ultimate load $F$ that caused them to occur.

14. Ultimate Load – Sample Problem

The concept of the occurrence of plastic hinges due to the limit loading of a beam will now be applied to the previous example problem. All forces acting on a free body diagram of the entire beam must first satisfy equilibrium. This gives rise to equation 1 that states that the sum of the support reactions, $F_1$ and $F_2$, must be equal to the applied load $F$. Equation 2 comes from summing all moments about the left end of the beam. Note that the moment reaction at the left end is set equal to the plastic moment, $M_p$, since a plastic hinge is required at this location if a plastic hinge exists at the location of the load $F$. These two equations contain three unknowns, the reactions $F_1$ and $F_2$ together with the unknown limit load $F$. Hence a third equation is required. This is obtained by recognizing the plastic moment at the location, $a$, of the load, $F$, can be written as $F_1$ times the distance $a$. This then establishes the value of this reaction permitting $F$ and $F_2$ to be determined from equations 1 and 2.
15. Problem –1 Solution

Equations 2 and 3 from the previous page can be combined and solved directly for $F$ in terms of the plastic moment $M_p$ and the geometry of the beam. This expression gives the value of the limit load $F$ that will produce plastic hinges at its position and the fixed end resulting in the shape of the deformation curve on the previous page. This model will not predict the magnitude of the deflection at point $a$. The assumption of unlimited material strain following yielding in the idealized material stress strain behavior results in the deflection being indeterminant. With $F$, the limit load, determined the problem is essentially solved. However, the remaining reaction $F_2$ can also now be determined as

\[
F_2 = \frac{M_p}{a} \left( \frac{2a}{c-a} \right)
\]

16. Exercise Problem - 2

In this exercise you are asked to use the reaction results of the previous problem solution to determine the value of the bending moment at the simply supported right end of the beam. If the solution is correct the bending moment should be zero. After you have completed your analysis click on the solution button for verification before proceeding on with the rest of the chapter.

(Solution on Page 242)
17. Problem – 1 Solution

The answer for the limit load in the previous problem is expressed in terms of the length of the beam and the location of the load $F$. This will now be examined in terms of the ratio of the distance $a$ to the length of the beam $c$. This is first made possible by rearranging the equation for $F$ into the dimensionless representation involving the ratio of $F$ times $c$ over $M_p$ and the ratio of $a$ to $c$. Now numerical values can be calculated for the dimensionless limit load. The table presents numerical values of the limit load parameter for values of dimensionless load location for increments of $a$ over $c$ of 0.1. It should be noted that if $a$ over $c$ is either 0 or 1 the load parameter is infinite. This corresponds to the load being over a support, which could result in a very large load if the support were strong enough. As $a$ over $c$ increases the load parameter first decreases and then increases. As it gets closer to the fixed end its values are higher than at the simply supported end as would be expected. Also its smallest value occurs just short of the center of the beam from the right end.

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<th>$Fc/M_p$</th>
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</table>

18. Exercise Problem - 3

In this exercise you are asked to use the load limit analysis process just presented to determine the limit load that can be carried by a simply supported beam as a function of the position of the force from one end. How does the result differ from the solution to the previous problem? After completing your analysis click on the solution button to check your result before proceeding on to the remainder of the chapter.

(Solution on Pages 242 and 243)
19. Sample Problem - 2

The fixed and simply supported beam will now be reanalyzed assuming the beam is subjected to a uniformly distributed loading system. Again the limit load, \( w \), is desired. The solution is begun by applying the equations of equilibrium to a free body diagram of the entire beam. Equation 1 satisfies vertical equilibrium while equation 2 comes from summing moments about the right end. A third equation is obtained by assuming that a plastic hinge occurs at some yet to be determined distance \( a \) from the left end and writing an equation summing the moments on a free body diagram of the section of length \( a \). These three equations involve four unknowns, \( F_1 \), \( F_2 \), \( w \) and \( a \). To obtain a deterministic solution it is necessary to solve for \( F_1 \), \( F_2 \) and \( w \) in terms of \( a \) and then to vary the value of \( a \) to obtain the minimum value of \( w \). This will be the limit load that produces the plastic hinge behavior.

20. Problem – 2 Solution

The second equilibrium equation is first solved for \( F_2 \) in terms of \( M_p \) and \( w \). This result is then substituted into equation 3, which is the plastic hinge equation, and introduces the parameter \( a \), the location of the plastic hinge. This equation must then be solved for \( w \) in terms of \( M_p \) and \( a \).
### Problem - 2 Solution

The last equation on the previous page is solved for w and put into a non dimensional form involving the load parameter \( w_c^2 \) squared over \( M_p \) as a function of the ratio of \( a \) over \( c \). Numerical values of the load parameter are now determined along the length of the beam at increments of \( a \) over \( c \) of 0.1. The table presents the results of these calculations. Note that \( a \) over \( c \) values of 0 and 1 are not included since a plastic hinge cannot occur over a simple support. Again the load parameter is seen to decrease and then increase as the position parameter increases. The minimum value of the load parameter from this table is 11.67 and occurs at \( a \) over \( c \) of 0.6. Thus the correct value of the limit load parameter is about 11.5 and the previously unknown hinge position occurs just a little past the center from the fixed end. This position appears quite reasonable. To obtain a more exact solution the position of the zero slope of the load parameter curve as a function of the hinge position parameter could be determined analytically.

<table>
<thead>
<tr>
<th>( a/c )</th>
<th>( w_c^2M_p )</th>
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</tr>
<tr>
<td>0.9</td>
<td>24.44</td>
</tr>
</tbody>
</table>

### Exercise Problem - 4

In this exercise you are asked to determine the limit load for a uniformly distributed load on a beam simply supported at each end. Then repeat the solution for a beam that is fixed at both ends. Compare the results. Do they appear reasonable? Check your answers by clicking on the solution button before going on to the rest of the chapter.

(Solution on Page 244)
23. Sample Problem – 3

The problem of determining the concentrated load required to collapse a circular ring will now be analyzed. In this solution it is assumed that the thickness of the ring is small compared to its radius so that simple bending theory can be applied to any cross section. It is then assumed that collapse requires the occurrence of four plastic hinges, two on the horizontal diameter and two on the vertical diameter. To determine the limit load \( F \) only one quarter of the ring need be analyzed due to the horizontal and vertical symmetry of the system. Vertical equilibrium of the quarter of the ring in the second quadrant requires that the half of the external vertical load at the top must be balance by an equal compressive axial force at the cut cross section on the horizontal diameter. At the same time plastic moments, \( M_p \), must also act at the horizontal cut cross section and the vertical cross section at the top.

24. Problem 3 - Solution

A hinge moment equation for the plastic bending moment can now be written for either the top or horizontal diameter cross sections. Both of these equations are exactly the same and give a final result of the limit load being equal to 4 times the plastic moment \( M_p \) divided by the ring radius \( R \). How would the problem of the limit load for a ring collapsed by both horizontal and vertical concentrated forces of the same magnitude be solved? Would it require more than four plastic hinges?
### Review Exercises

1. The magnitude of the plastic moment does not depend on the shape of the beam cross section.  
   - True  
   - False
2. On what type of static load system will the limit load analysis work?  
   - Determinate  
   - Indeterminate  
   - Either one
3. A plastic hinge can only occur at points of maximum bending moment along the beam?  
   - True  
   - False
4. The load limit analysis can be applied to what kind of material?  
   - Brittle  
   - Ductile  
   - Both

### Off line Exercises

1. A beam of length "c" is built in at both ends and is subjected to a single concentrated vertical load "F". Using limit analysis determine the maximum value of the load as a function of its position "a" from one end.
2. A circular geometry hook of radius "R" is loaded as shown. Use limit analysis to determine the maximum value of "F".

### 25. Review Exercise

In the review exercise simply click on the appropriate answer. You will be provided with an immediate feedback. If you select the wrong answer try again. The hot words in the questions will link back to the page in the chapter that deals with the issue raised in that question. When you have completed the quiz go on to the next page.

### 26. Off Line Exercises

Use the limit load analysis procedure presented in chapter 10 to solve these two problems. When you are finished with these problem statements click on the main menu or exit buttons to leave the chapter.
Chapter 10

Limit Loading

Problem Solutions

Screen Titles

Problem 1 Solution
Problem 1 Solution (cont..)
Problem 2 Solution
Problem 3 Solution
Problem 3 Solution (cont..)
Problem 3 Solution (cont..)
Problem 4 Solution
Problem 4 Solution (cont.)
1. Problem 1 - Solution

The centroid of the top half of the circular cross section is located at 2R over Pi above the horizontal diameter. This is the point through which the resultant axial force for the top half of the cross section acts. Therefore, the plastic moment is equal to the couple of the F resultants which is F times 2 times 2R over Pi. With F equal to sigma y times Pi R squared over 2 the final expression for the plastic moment, Mp, is given by 2 sigma y times R cubed.

2. Problem 1 - Solution (cont.)

This plastic moment is now to be compared to the elastic moment at the onset of yielding. Begin with the expression for stress as M c over I. For a circular cross section I is equal to Pi R fourth over 4 and c is equal to R. Solving for M and substituting sigma y for sigma together with I and c in terms of R results in Me equal to sigma y R cubed times the quantity Pi over 4. Thus the ratio of Mp to Me becomes 8 over Pi or 2.55. This is significantly greater than the value of this ratio for a rectangular cross section. The reason for this is that more of the material of the cross section is closer to the horizontal axis where the constant yield stress has more of an effect compared to the linear distribution. When you are finished with this solution click on the return button to go back to chapter 10.
3. Problem 2 – Solution

Summing moments about the right end of the beam consists of three contributions. The first is the plastic moment, which is considered positive. The second is a negative contribution of $F_2$ times the length of the beam and the third is a positive contribution of $F$ times its moment arm $a$. Substituting the determined values of $F_2$ and $F$ into the equation permits $M_p$ to be divided out as a common factor. Combining the remaining terms over a common denominator results in a numerator in which all the terms cancel each other. Thus the left side of the equation is zero which satisfies the condition that the right end is simply supported.

4. Problem 3- Solution

A simple supported beam acted on by a concentrated load will only possess one plastic hinge under the location of the vertical load. This is where the bending moment diagram will have its largest value. This plastic moment will be produced by the two end reactions on the beam times the moment arm from their location to where the vertical force $F$ acts. This is expressed analytically by the first two equations at the top of the page. With $F_1$ and $F_2$ known the applied limit load $F$ can be determined as their sum. This result in an equation for $F$ in terms of the plastic moment $M_p$ and the geometry of the beam and loading represented by $c$ and $a$. For a numerical comparison with the solution for the fixed – simply supported beam already solved this final expression is put into a similar dimensionless form.
5. Problem 3 – Solution (cont.)

Substituting numerical values of the dimensionless load location, \( a/c \), at increments of 0.1 from the end to the center of the beam gives the listed table of values for the dimensionless load parameter, \( F \) times \( c/M_p \). At \( a/c \) equal to zero the load is over the end support, which is represented mathematically by infinity and is not listed in the table. Note that the load parameter decreases from near the end support to the center by almost a factor of three. For the second half of the beam the values would be repeated since this load function in terms of its location parameter will be symmetric about the center of the beam.

6. Problem 3 – Solution (cont.)

This page compares the results of the solution of the concentrated load on a simply supported beam at both ends to a concentrated load on a beam that is fixed at one end and simply supported at the other. Note that at all locations the fixed beam will support a greater load than the one that is simply supported at both ends, as one would expect. Also note that even at the center of both beams the SS-Fixed beam will carry 50% more load than the beam with simple supports at both ends. All of these effects are due to the clamped end support, which provides for greater strength. When you have finished with this page click on the return button to go on to the next page in chapter 10.
7. Problem 4 – Solution

A simply supported beam carrying a uniform distributed load, w, will result in the occurrence of a plastic hinge at its center from simple considerations of symmetry. In this instance the two support reactions are equal and given by one half the total load represented by w times c. Either of the support reactions can now be used to calculate the value of the plastic moment at the center of the beam. The contribution of the support reaction given by F times c over 2 minus the contribution of the distributed load which in terms of its resultant is w times c over 2 times the moment arm c over 4 must be equal to M_p. Solving for w gives 8M_p divided by c squared.

8. Problem 4 – Solution (cont.)

Now consider the beam from the previous page to be fixed at both ends under the same distributed load, w. In this case there must be a plastic hinge at both ends as well as in the center. From vertical equilibrium and symmetry the two support reactions are again equal and have a magnitude of wc over 2. The plastic hinge equation is similar to that developed on the previous page with the exception that a negative moment M_p must be included on the left side representing the plastic moment reaction at the fixed end of the beam. Solving this revised hinge moment equation gives a final value for w of 16 M_p over c squared or twice the limit load for the simply supported beam. When finished with this solution click on the return button to go back to the next page in chapter 10.